UNIVERSITY OF WATERLOO
FACULTY OF ENGINEERING
Department of Electrical &
Computer Engineering

ECE 204 *Numerical methods*

Approximating solutions to Laplace's equation

Douglas Wilhelm Harder, LEL, M.Math.
dwharder@uwaterloo.ca
dwharder@gmail.com

Introduction

- In this topic, we will
  - Introduce Laplace's equation
  - Discuss solutions to Laplace's equation in one dimesion
  - Convert the equation to a finite-difference equation in two and three dimensions
  - Discuss how to create a system of linear equations to find an approximation to the solution
  - Look at a number of examples both in two and three dimensions

Laplace's equation

- *Laplace's equation* is reasonably straight-forward:

$$\nabla^2 u(\mathbf{x}) = 0$$

  - This says that the sum of concavities at every point is zero
  - Forces are proportional to acceleration, so this essentially says the forces at each point are balanced
  - If all the forces are balanced, there will be no change to velocity
  - If the system is not moving, it will remain so

Laplace's equation

- In one dimension, Laplace's equation says

$$\frac{\partial^2}{\partial x^2} u(x) = 0$$

  - This means that $u(x)$ can be at most a straight line $ax + b$

Laplace's equation

- A solution is decided by boundary values:

$$\frac{\partial^2}{\partial x^2} u(x) = 0$$

  - If $u(a) = u_a$ and $u(b) = u_b$, we have a trivial unique solution:

$$u(x) = u_a + (x - a)\frac{u_b - u_a}{b - a}$$

  - If $u(a) = u_a$ and $u^{(1)}(b) = u_b^{(1)}$, we have another trivial unique solution:

$$u(x) = u_a + (x - a)u_b^{(1)}$$

  - If $u^{(1)}(a) = u_a^{(1)}$ and $u^{(1)}(b) = u_b^{(1)}$, we either have no solutions or infinitely many solutions
    - Sound familiar?

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Laplace's equation

- Notice that for if a solution satisfies Laplace's equation,
  it is also a steady-state solution for both the heat equation and the wave equation

$$\frac{\partial^2}{\partial x^2} u(x) = 0$$

  - A solution to the heat equation converges to a solution of Laplace's equation
  - A solution to the wave equation oscillates around a solution to Laplace's equation

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Finite-difference approximation

- In two and three dimensions, it becomes more interesting:

$$\frac{\partial^2}{\partial x^2}u(x,y)+\frac{\partial^2}{\partial y^2}u(x,y)=0$$

$$\frac{\partial^2}{\partial x^2}u(x,y,z)+\frac{\partial^2}{\partial y^2}u(x,y,z)+\frac{\partial^2}{\partial z^2}u(x,y,z)=0$$

  - In two dimensions,
    this requires a region in the plane with a specified boundary
  - In three dimensions,
    this requires a volume with a specified boundary
  - A Dirichlet (fixed) or Neumann (fixed derivative) must be specified at each point on the boundary

Finite-difference approximation

- Applications of Laplace's equation include:
  - If all the walls in a room are either insulated or at fixed (and possibly different temperatures), the temperatures throughout the room will converge to a solution to Laplace's equation
    - The gradient would specify the direction of maximum increase in temperature at any point
  - If each point on a boundary either has a specified fixed voltage or is insulated, and the region is devoid of any charges, the potential throughout the region is specified by Laplace's equation
    - The gradient of the solution specifies the direction that a test charge would follow

Approximating solutions

- As before, substitute our divided-difference approximations for the second partial derivative into the equation

$$\frac{u(x+h,y)-2u(x,y)+u(x-h,y)}{h^2}+\frac{u(x,y+h)-2u(x,y)+u(x,y-h)}{h^2}=0$$

  – Multiply both sides by $-h^2$ and collect:

$$4u(x,y)-u(x+h,y)-u(x-h,y)-u(x,y+h)-u(x,y-h)=0$$

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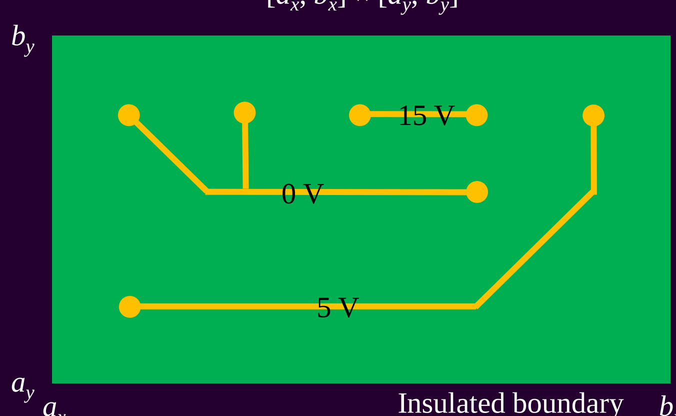
The wave equation

Approximating solutions

- Suppose that a region in the plane is entirely contained within the rectangle defined by

$$[a_x, b_x] \times [a_y, b_y]$$



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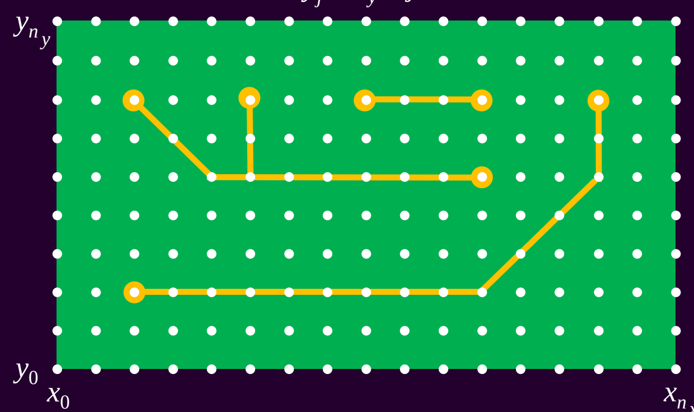
Approximating solutions

- Divide the region into $n_x \times n_y$ squares and define

$$x_i = a_x + ih$$
$$y_j = a_y + jh$$



Finite-difference approximation

- Now substitute x_i and y_j into our equation

$$4u(x_i, y_j) - u(x_i + h, y_j) - u(x_i - h, y_j) - u(x_i, y_j + h) - u(x_i, y_j - h) = 0$$

- Next, recognize that $x_i \pm h = x_{i\pm1}$ and $y_j \pm h = y_{j\pm1}$,

$$4u(x_i, y_j) - u(x_{i+1}, y_j) - u(x_{i-1}, y_j) - u(x_i, y_{j+1}) - u(x_i, y_{j-1}) = 0$$

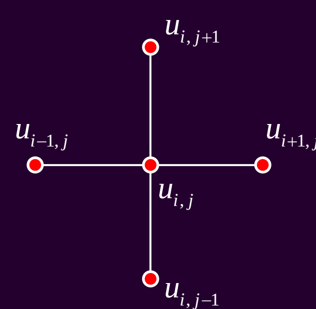
- We don't know what $u(x_i, y_j)$ is, so we will try to approximate it with an unknown $u_{i,j}$

$$4u_{i,j} - u_{i+1,j} - u_{i-1,j} - u_{i,j+1} - u_{i,j-1} = 0$$

Finite-difference approximation

- We can visualize this as follows:

$$4u_{i,j} - u_{i+1,j} - u_{i-1,j} - u_{i,j+1} - u_{i,j-1} = 0$$



Finite-difference approximation

- Note what this says:

$$4u_{i,j} - u_{i+1,j} - u_{i-1,j} - u_{i,j+1} - u_{i,j-1} = 0$$

- Rewrite this as:

$$4u_{i,j} = u_{i+1,j} + u_{i-1,j} + u_{i,j+1} + u_{i,j-1}$$

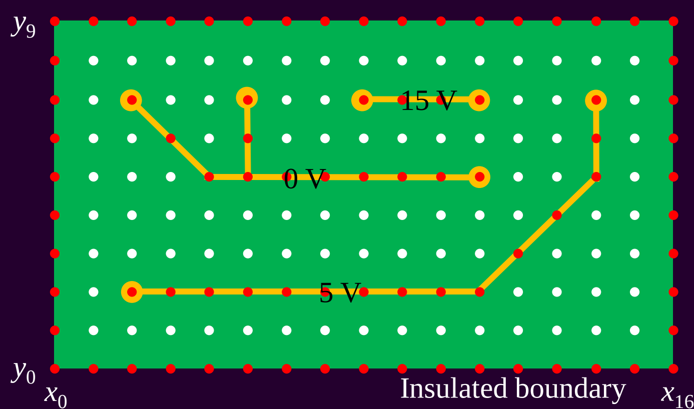
- Now, divide by four:

$$u_{i,j} = \frac{u_{i+1,j} + u_{i-1,j} + u_{i,j+1} + u_{i,j-1}}{4}$$

  – If u satisfies Laplace's equation,
    then $u(x_i, y_j)$ will be the average of the points around it

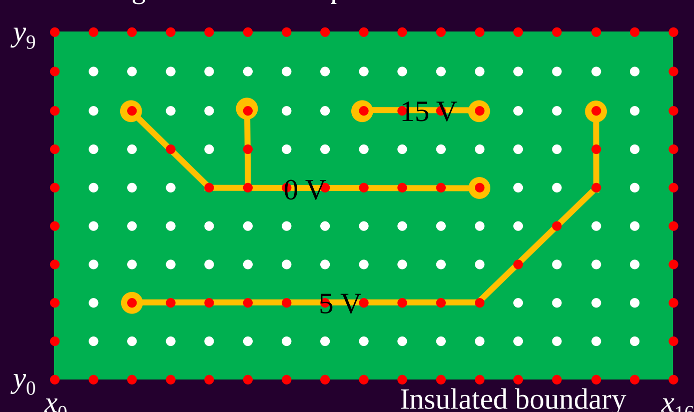
Finite-difference approximation

- Now, if (x_i, y_j) is a boundary point,
  its value or the slope at that point is already given
  - Each point in red has a boundary value



Finite-difference approximation

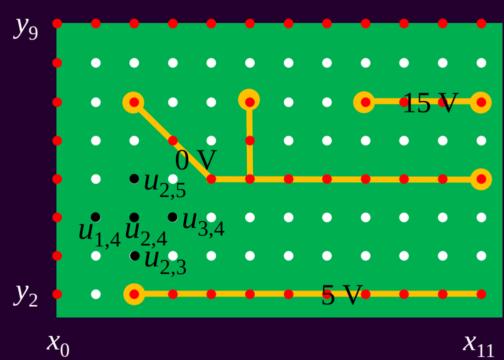
- Now, if (x_i, y_j) is a boundary point,
  its value or the slope at that point is already given
  - The voltage at each white point is unknown…



Finite-difference approximation

- Let's zoom in at a few points: $u_{2,4}$
  - The finite-difference equation says:

$$4u_{2,4} - u_{3,4} - u_{1,4} - u_{2,5} - u_{2,3} = 0$$



The wave equation

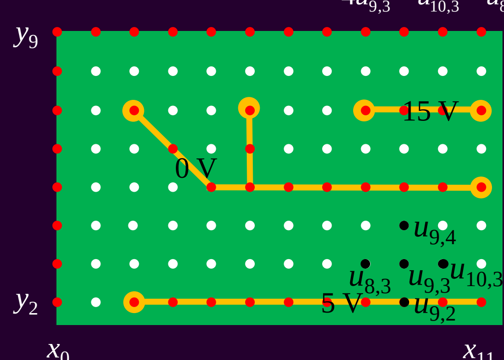
Finite-difference approximation

- Let's zoom in at a few points: $u_{9,3}$
  - The finite-difference equation says:

$$4u_{9,3} - u_{10,3} - u_{8,3} - u_{9,4} - u_{9,2} = 0$$

$$4u_{9,3} - u_{10,3} - u_{8,3} - u_{9,4} - 5 = 0$$

$$4u_{9,3} - u_{10,3} - u_{8,3} - u_{9,4} = 5$$



The wave equation

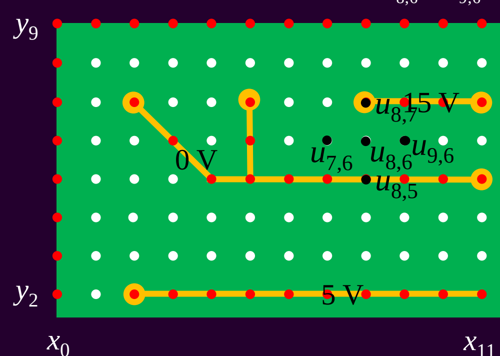
Finite-difference approximation

- Let's zoom in at a few points: $u_{8,6}$
  - The finite-difference equation says:

$$4u_{8,6} - u_{9,6} - u_{7,6} - u_{8,7} - u_{8,5} = 0$$

$$4u_{8,6} - u_{9,6} - u_{7,6} - 15 - 0 = 0$$

$$4u_{8,6} - u_{9,6} - u_{7,6} = 15$$



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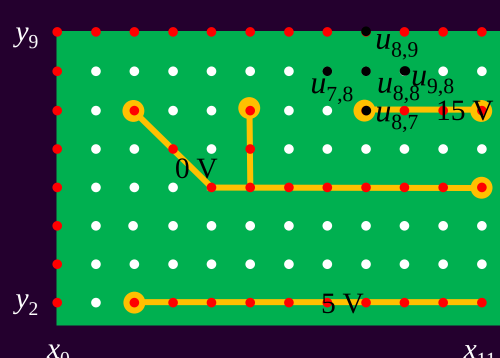
The wave equation

Finite-difference approximation

- Let's zoom in at a few points: $u_{8,8}$
  - The finite-difference equation says:

$$4u_{8,8} - u_{9,8} - u_{7,8} - u_{8,9} - u_{8,7} = 0$$

$$4u_{8,8} - u_{9,8} - u_{7,8} - 0 - 15 = 0$$



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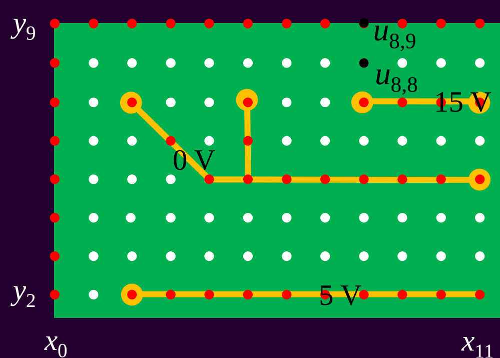
Finite-difference approximation

- How do we deal with an insulated condition?
  - If the derivative at an insulated boundary point is zero,

$$\text{then } \frac{u_{8,9} - u_{8,8}}{h} = 0 \text{ and so } u_{8,9} - u_{8,8} = 0 \text{ or } u_{8,9} = u_{8,8}$$

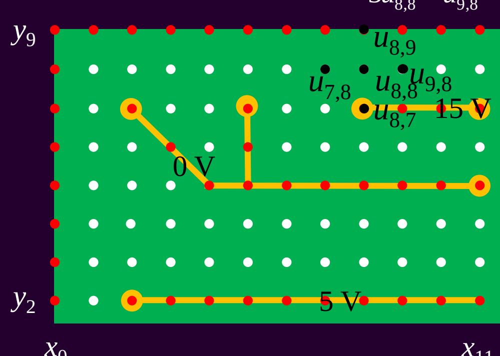


Finite-difference approximation

- Let's zoom in at a few points: $u_{8,8}$
  - Substituting in the value $u_{8,8}$ for the insulated point, we have

$$4u_{8,8} - u_{9,8} - u_{7,8} \boxed{-u_{8,8}} - 15 = 0$$

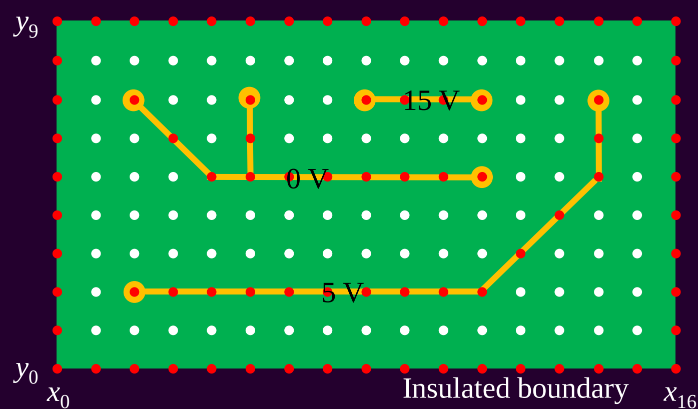
$$3u_{8,8} - u_{9,8} - u_{7,8} = 15$$



The wave equation

Finite-difference approximation

- There are 89 unknowns in this layout
  - If we repeated the previous process at each of these 89 points, we would have 89 linear equations in 89 unknowns



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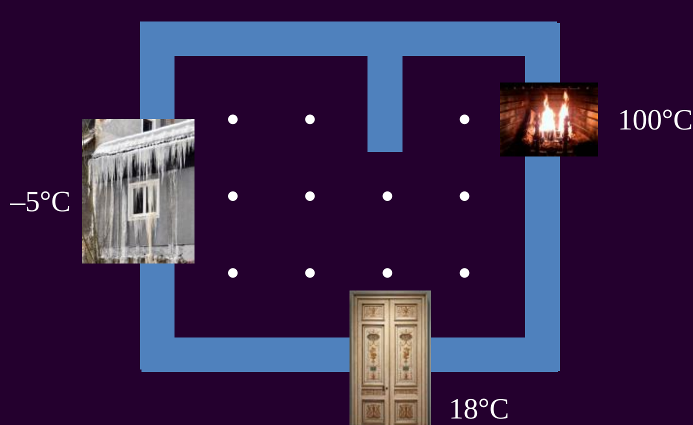
The wave equation

A simpler example

- Here is an easier example



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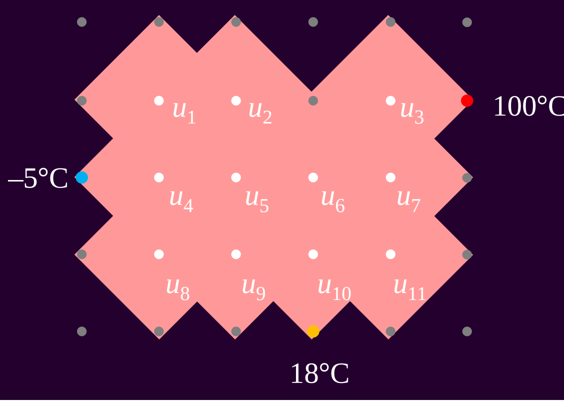
A simpler example

- There are eleven interior points

$$
\begin{aligned}
&2u_1 - u_2 - u_4 = 0\\
&2u_2 - u_1 - u_5 = 0\\
&2u_3 - u_7 = 100\\
&4u_4 - u_1 - u_5 - u_8 = -5\\
&4u_5 - u_2 - u_4 - u_6 - u_9 = 0\\
&3u_6 - u_5 - u_7 - u_{10} = 0\\
&3u_7 - u_3 - u_6 - u_{11} = 0\\
&2u_8 - u_4 - u_9 = 0\\
&3u_9 - u_5 - u_8 - u_{10} = 0\\
&4u_{10} - u_6 - u_9 - u_{11} = 18\\
&2u_{11} - u_7 - u_{10} = 0
\end{aligned}
$$



The wave equation

A simpler example

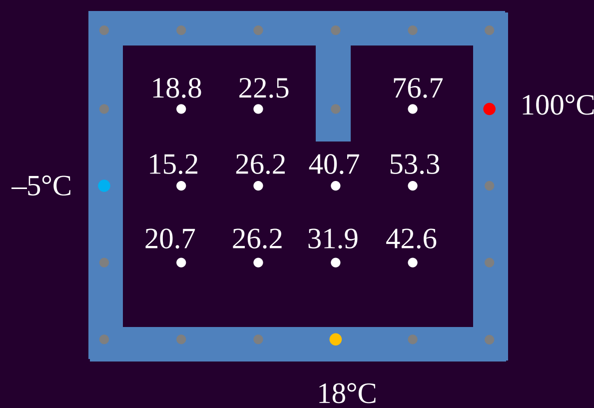
- Here is the system of linear equations:

2	–1		–1								0
–1	2			–1							0
		2				–1					100
–1			4	–1			–1				–5
	–1		–1	4	–1			–1			0
				–1	3	–1			–1		0
		–1			–1	3				–1	0
			–1				2	–1			0
				–1			–1	3	–1		0
					–1			–1	4	–1	18
						–1			–1	2	0

$$\mathbf{u} = \begin{pmatrix} 18.8 \\ 22.5 \\ 76.7 \\ 15.2 \\ 26.2 \\ 40.7 \\ 53.3 \\ 20.7 \\ 26.2 \\ 31.9 \\ 42.6 \end{pmatrix}$$

A simpler example

- Copying the solutions back to our room



$$\mathbf{u} = \begin{pmatrix} 18.8 \\ 22.5 \\ 76.7 \\ 15.2 \\ 26.2 \\ 40.7 \\ 53.3 \\ 20.7 \\ 26.2 \\ 31.9 \\ 42.6 \end{pmatrix}$$

Three dimensions

- With a little effort, you should realize that the finite-difference approximation of Laplace's equation in three dimensions is:

$$\frac{\partial^2}{\partial x^2}u(x,y,z)+\frac{\partial^2}{\partial y^2}u(x,y,z)+\frac{\partial^2}{\partial z^2}u(x,y,z)=0$$

$$6u_{i,j,k}-u_{i+1,j,k}-u_{i-1,j,k}-u_{i,j+1,k}-u_{i,j-1,k}-u_{i,j,k+1}-u_{i,j,k-1}=0$$

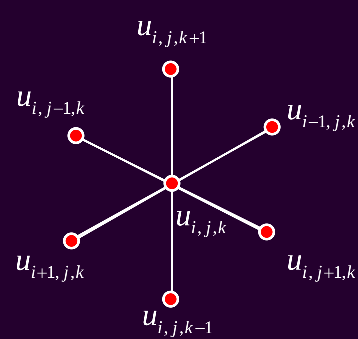
- As before, the value at each point is the average of the six points surrounding it:

$$u_{i,j,k}=\frac{u_{i+1,j,k}+u_{i-1,j,k}+u_{i,j+1,k}+u_{i,j-1,k}+u_{i,j,k+1}+u_{i,j,k-1}}{6}$$

Three dimensions

- We can visualize this as follows:

$$u_{i,j,k} = \frac{u_{i+1,j,k} + u_{i-1,j,k} + u_{i,j+1,k} + u_{i,j-1,k} + u_{i,j,k+1} + u_{i,j,k-1}}{6}$$



Three dimensions

- Suppose we find a box in $\mathbf{R}^3$ is such that our region is entirely contained in

$$[a_x, b_x] \times [a_y, b_y] \times [a_z, b_z]$$

  - This region should be such that it can be divided into cubes of dimensions h^3 so $b_x - a_x = n_x h$, etc.
  - Thus, we can define

$$x_i = a_x + ih$$
$$y_j = a_y + jh$$
$$z_k = a_z + kh$$

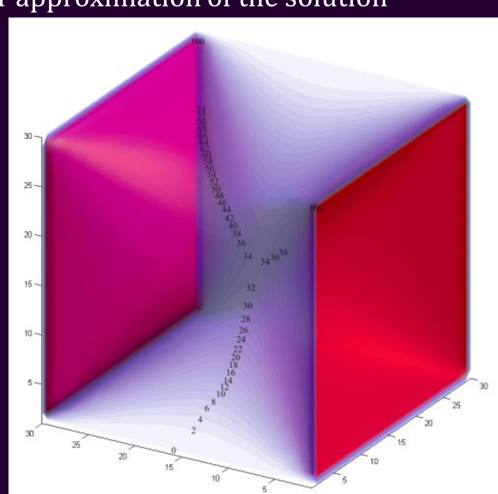
  - We will approximate $u(x_i, y_j, z_k)$ by $u_{i,j,k}$
  - As in two dimensions, this will define a system of linear equations which we can solve

Example

- Here is one example of such a problem:
  - Consider a cube such that two opposite faces are at either 100 V or 100°C and the four remaining sides are at 0 V or 0°C
  - We could choose an h so that $n_x = n_y = n_z = 30$

Example

- Here is our approximation of the solution



The wave equation

Example

- Here is another example of such a problem:
  - Consider a cube such that all sides are at 0 V or 0°C and there are three point sources within that are at 100 V or 100°C
  - Again, we could choose an h so that $n_x = n_y = n_z = 30$

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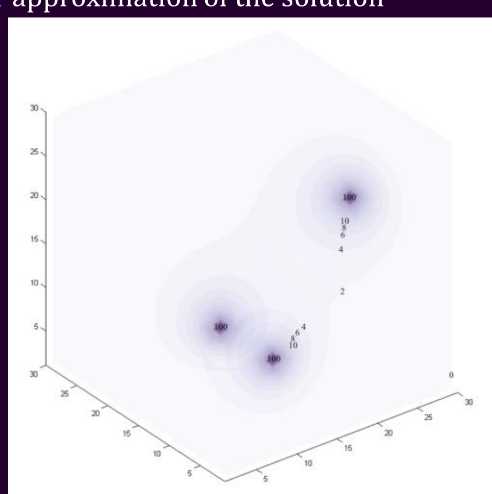
The wave equation

Example

- Here is our approximation of the solution



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Example

- Here is another example of such a problem:
  - Consider a cube such that one side is at 0 V or 0°C and the remaining five sides are insulated
  - There is a point source at the center at 100 V or 100°C
  - Again, we could choose an h so that $n_x = n_y = n_z = 14$

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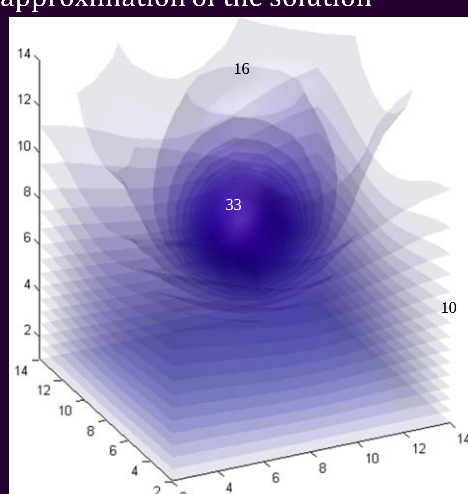
Example

- Here is our approximation of the solution



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Example

- Thus, for most unknown points, the linear equation would be:

$$6u_{i,j,k} - u_{i+1,j,k} - u_{i-1,j,k} - u_{i,j+1,k} - u_{i,j-1,k} - u_{i,j,k+1} - u_{i,j,k-1} = 0$$

  – If one of the six neighboring points is the source at 100, the equation would be

$$6u_{i,j,k} - u_{i+1,j,k} - u_{i-1,j,k} - u_{i,j+1,k} - 100 - u_{i,j,k+1} - u_{i,j,k-1} = 0$$

$$6u_{i,j,k} - u_{i+1,j,k} - u_{i-1,j,k} - u_{i,j+1,k} - u_{i,j,k+1} - u_{i,j,k-1} = 100$$

  – If one of the six neighboring points is the wall kept at 0, the equation would be

$$6u_{i,j,k} - u_{i+1,j,k} - u_{i-1,j,k} - u_{i,j+1,k} - u_{i,j-1,k} - u_{i,j,k+1} - 0 = 0$$

$$6u_{i,j,k} - u_{i+1,j,k} - u_{i-1,j,k} - u_{i,j+1,k} - u_{i,j-1,k} - u_{i,j,k+1} = 0$$

  – If one of the six neighboring points is an insulated point, the equation would be

$$5u_{i,j,k} - u_{i+1,j,k} - u_{i,j+1,k} - u_{i,j-1,k} - u_{i,j,k+1} - u_{i,j,k-1} = 0$$

Summary

- Following this topic, you now
  – Understand Laplace's equation and its application
  – Know how to convert Laplace's equation into a finite-difference equation
  – Know how to break a region into a grid of points
  – Have seen four examples including Dirichlet (fixed) and insulated boundary conditions

References

[1] https://en.wikipedia.org/wiki/Laplace%27s_equation

Acknowledgments

None so far.

Colophon

These slides were prepared using the Cambria typeface. Mathematical equations use Times New Roman, and source code is presented using `Consolas`.
Mathematical equations are prepared in MathType by Design Science, Inc.
Examples may be formulated and checked using Maple by Maplesoft, Inc.

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for more information.

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